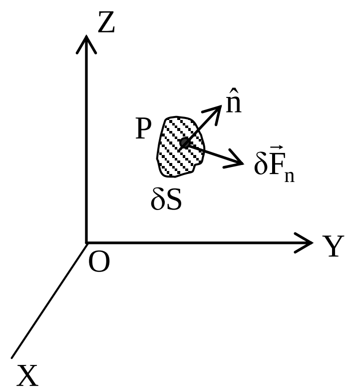
UNIT –V

1. Stress Components in a Real Fluid

Let δS be a small rigid plane area inserted at a point P in a viscous fluid. Cartesian co-ordinates (x, y, z) are referred to a set of fixed axes OX, OY, OZ. Suppose that $\delta \overline{F}_n$ is the force exerted by the moving fluid on one side of δS, the unit vector $\hat{n}$ being taken to specify the normal at P to δS on this side. We know that in the case of an inviscit fluid, $\delta \overline{F}_n$ is aligned with $\hat{n}$. For a viscous fluid, however, frictional forces are called into play between the fluid and the surface so that $\delta \overline{F}_n$ will also have a component tangential to δS. We suppose the Cartesian components of $\delta \overline{F}_n$ to be (δF_{nx}, δF_{ny}, δF_{nz}) so that



$$\delta \overline{F}_n = \delta F_{nx}\,\hat{i} + \delta F_{ny}\,\hat{j} + \delta F_{nz}\,\hat{k}.$$

Then the components of stress parallel to the axes are defined to be σ_{nx}, σ_{ny}, σ_{nz}, where

$$\sigma_{nx} = \underset{\delta S \to 0}{\mathrm{lt}} \frac{\delta F_{nx}}{\delta S} = \frac{dF_{nx}}{dS},$$

$$\sigma_{ny} = \underset{\delta S \to 0}{\mathrm{lt}} \frac{\delta F_{ny}}{\delta S} = \frac{dF_{ny}}{dS},$$

$$\sigma_{nz} = \underset{\delta S \to 0}{\mathrm{lt}} \frac{\delta F_{nz}}{\delta S} = \frac{dF_{nz}}{dS}.$$

In the components σ_{nx}, σ_{ny}, σ_{nz}, the first suffix n denotes the direction of the normal to the elemental plane δS whereas the second suffix x or y or z denotes the direction in which the component is measured.

If we identify $\hat{n}$ in turn with the unit vectors $\hat{i}, \hat{j}, \hat{k}$ in $(\overline{OX}), (\overline{OY}), (\overline{OZ})$, which is achieved by suitably re-orientating δS, we obtain the following three sets of stress components

$$\sigma_{xx}, \quad \sigma_{xy}, \quad \sigma_{xz};$$

$$\sigma_{yx}, \quad \sigma_{yy}, \quad \sigma_{yz};$$

$$\sigma_{zx}, \quad \sigma_{zy}, \quad \sigma_{zz}.$$

The diagonal elements σ_{xx}, σ_{yy}, σ_{zz} of this array are called normal or direct stresses. The remaining six elements are called shearing stresses. For an inviscid fluid, we have

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p$$

$$\sigma_{xy} = \sigma_{xz} = \sigma_{yx} = \sigma_{yz} = \sigma_{zx} = \sigma_{zy} = 0$$

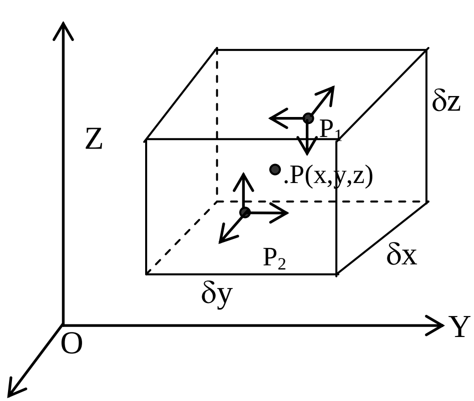
Here, we consider the normal stresses as positive when they are tensile and negative when they are compressive, so that p is the hydrostatic pressure. The matrix

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \qquad (1)$$

is called the stress matrix. If its components are known, we can calculate the total forces on any area at any chosen point. The quantities $\sigma_{ij}(i, j = x, y, z)$ are called the components of the stress tensor whose matrix is of the form (1). Further we observe that σ_{ij} is a tensor of order two.

2. Relation Between Rectangular (Cartesian) Components of Stress

Let us consider the motion of a small rectangular parallelopiped of viscous fluid, its centre being P(x, y, z) and its edges of lengths δx, δy, δz, parallel to fixed Cartesian axes, as shown in the figure.



X

Let ρ be the density of the fluid. The mass $\rho\delta x\ \delta y\ \delta z$ of the fluid element remains constant and the element is presumed to move alongwith the fluid. In the figure, the points P_1 and P_2 have been taken on the centre of the faces so that they have co-ordinates $\left(x-\frac{\delta x}{2}, y, z\right)$ and $\left(x+\frac{\delta x}{2}, y, z\right)$ respectively.

At P(x, y, z), the force components parallel to $\overline{OX}, \overline{OY}, \overline{OZ}$ on the surface area $\delta y.\ \delta z$ through P and having $\hat{i}$ as unit normal, are

$$(\sigma_{xx}\delta y\ \delta z,\quad \sigma_{xy}\ \delta y\delta z,\quad \sigma_{xz}\ \delta y\delta z)$$

At $P_2\left(x+\frac{\delta x}{2}, y, z\right)$, since $\hat{i}$ is the unit normal measured outwards from the fluid, the corresponding force components across the parallel plane of area $\delta y\delta z$, are

$$\left[\left\{\sigma_{xx}+\frac{\delta x}{2}\left(\frac{\partial\sigma_{xx}}{\partial x}\right)\right\}\delta y\delta z, \left\{\sigma_{xy}+\frac{\delta x}{2}\left(\frac{\partial\sigma_{xy}}{\partial x}\right)\right\}\delta y\delta z, \left\{\sigma_{xz}+\frac{\delta x}{2}\left(\frac{\partial\sigma_{xz}}{\partial x}\right)\right\}\delta y\delta z\right].$$

For the parallel plane through $P_1\left(x-\frac{\delta x}{2}, y, z\right)$, since $-\hat{i}$ is the unit normal drawn outwards from the fluid element, the corresponding components are

$$\left[-\left\{\sigma_{xx}-\frac{\delta x}{2}\left(\frac{\partial\sigma_{xx}}{\partial x}\right)\right\}\delta y\delta z, -\left\{\sigma_{xy}-\frac{\delta x}{2}\left(\frac{\partial\sigma_{xy}}{\partial x}\right)\right\}\delta y\delta z, -\left\{\sigma_{xz}-\frac{\delta x}{2}\left(\frac{\partial\sigma_{xz}}{\partial x}\right)\right\}\delta y\delta z\right]$$

The forces on the parallel planes through P_1 and P_2 are equivalent to a single force at P with components

$$\left[\frac{\partial\sigma_{xx}}{\partial x}, \frac{\partial\sigma_{xy}}{\partial x}, \frac{\partial\sigma_{xz}}{\partial x}\right]\delta x\ \delta y\ \delta z$$

together with couples whose moments (upto third order terms) are

$$\begin{cases} -\sigma_{xz}\ \delta x\ \delta y\ \delta z \text{ about } Oy, \\ \sigma_{xy}\ \delta x\delta y\delta z \text{ about } Oz. \end{cases}$$

Similarly, the pair of faces perpendicular to the y axis give a force at P having components

$$\left[\frac{\partial\sigma_{yx}}{\partial y}, \frac{\partial\sigma_{yy}}{\partial y}, \frac{\partial\sigma_{yz}}{\partial y}\right]\delta x\ \delta y\ \delta z$$

together with couples of moments

$$\begin{cases} -\sigma_{yx}\ \delta x\ \delta y\ \delta z \text{ about } Oz, \\ \sigma_{yz}\ \delta x\delta y\delta z \text{ about } Ox. \end{cases}$$

The pair of faces perpendicular to the z-axis give a force at P having components

$$\left[\frac{\partial\sigma_{zx}}{\partial z}, \frac{\partial\sigma_{zy}}{\partial z}, \frac{\partial\sigma_{zz}}{\partial z}\right]\delta x\ \delta y\ \delta z$$

together with couples of moments

$$\begin{cases} -\sigma_{zy}\ \delta x\ \delta y\ \delta z \text{ about } Ox, \\ \sigma_{zx}\ \delta x\delta y\delta z \text{ about } Oy. \end{cases}$$

Combining the surface forces of all six faces of the parallelopiped, we observe that they reduce to a single force at P having components

$$\left[\left(\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{yx}}{\partial y}+\frac{\partial\sigma_{zx}}{\partial z}\right),\left(\frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{zy}}{\partial z}\right),\left(\frac{\partial\sigma_{xz}}{\partial x}+\frac{\partial\sigma_{yz}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}\right)\right]\delta x\ \delta y\ \delta z,$$

together with a vector couple having Cartesian components

$$[(\sigma_{yz} - \sigma_{zy}), (\sigma_{zx} - \sigma_{xz}), (\sigma_{xy} - \sigma_{yx})]\ \delta x\ \delta y\ \delta z.$$

Now, suppose the external body forces acting at P are [X, Y, Z] per unit mass, so that the total body force on the element has components [X, Y, Z] $\rho\ \delta x\ \delta y\ \delta z$. Let us take moments about $\hat{i}$ –direction through P. Then, we have

Total moment of forces = Moment of inertia about axis × Angular acceleration

i.e. $(\sigma_{yz}-\sigma_{zy})\ \delta x\ \delta y\ \delta z$ + terms of 4th order in δx, $\delta y\ \delta z$ = terms of 5th order in δx, δy, δz.

Thus, to the third order of smallness in δx, δy, δz, we obtain

$$(\sigma_{yz} - \sigma_{zy})\, \delta x\, \delta y\, \delta z = 0$$

Hence, as the considered fluid element becomes vanishingly small, we obtain

$$\sigma_{yz} = \sigma_{zy}.$$

Similarly, we get

$$\sigma_{zx} = \sigma_{xz}, \qquad \sigma_{xy} = \sigma_{yx}$$

Thus, the stress matrix is diagonally symmetric and contains only six unknowns. In other words, we have proved that

$$\sigma_{ij} = \sigma_{ji},\ (i, j = x, y, z)$$

i.e. σ_{ij} is symmetric.

In fact, σ_{ij} is a symmetric second order Cartesian tensor.

3. Transnational Motion of Fluid Element. Considering the surface forces and body forces, we note (from the previous article) that the total force component in the $\hat{i}$ –direction, acting on the fluid element at point P(x, y, z), is

$$\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}\right)\delta x\, \delta y\, \delta z + X\, \rho\delta x\, \delta y\, \delta z \qquad (1)$$

where (X, Y, Z) is the body force per unit mass and ρ being the density of the viscous fluid. As the mass $\rho\, \delta x\, \delta y\, \delta z$ is considered constant, if $\bar{q} = (u, v, w)$ be the velocity of point P at time t, then the equation of motion in the $\hat{i}$ –direction is

$$\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}\right)\delta x\, \delta y\, \delta z + \rho\, X\, \delta x\, \delta y\, \delta z = (\rho\delta x\, \delta y\, \delta z)\frac{du}{dt}$$

or
$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + \rho X = \rho \frac{du}{dt} \qquad (2)$$

If u = u(x, y, z, t), then

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \text{ where } \frac{d}{dt} \equiv \frac{\partial}{\partial t} + \bar{q}\cdot\nabla$$

Thus, (2) becomes

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=X+\frac{1}{\rho}\left(\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{yx}}{\partial y}+\frac{\partial \sigma_{zx}}{\partial z}\right) \quad (3)$$

Similarly the equations of motion in $\hat{j}$ and $\hat{k}$ directions are

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=Y+\frac{1}{\rho}\left(\frac{\partial \sigma_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \sigma_{zy}}{\partial z}\right) \quad (4)$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=Z+\frac{1}{\rho}\left(\frac{\partial \sigma_{xz}}{\partial x}+\frac{\partial \sigma_{yz}}{\partial y}+\frac{\partial \sigma_{zz}}{\partial z}\right) \quad (5)$$

Equations (3), (4), (5) provide the equations of motion of the fluid element at P(x, y, z).

In tensor form, if the co-ordinates are x_i, the velocity components u_i, the body force components X_i, where i = 1, 2, 3, the equations of motion can be expressed as

$$\frac{\partial u_i}{\partial t}+u_j\, u_{i,j}=X_i+\frac{1}{\rho}\,\sigma_{ji,j} \text{ (i, j = 1, 2, 3).}$$

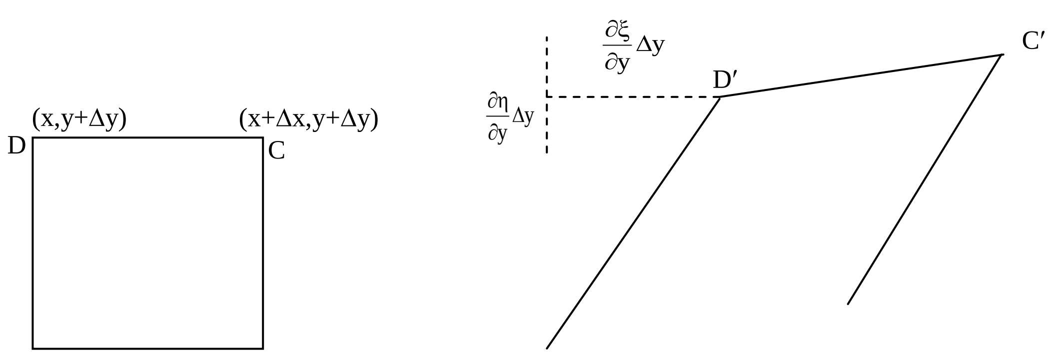
4. Nature of Strains (Rates of Strain)

The change in the relative position of the parts of the body under some force, is termed as **deformation.** By Hooke's law, the stress is proportional to strain in case of elastic bodies, while in case of non-elastic bodies the stress is proportional to the **rate of strain.**

Strain is of two kinds, the normal and the shearing. The ratio of change in length to the original length of a line element is called **normal (or direct) strain.** The shearing strain measures the change in angle between two line elements from the natural state to some standard state. We shall consider two dimensional case and then extend it to three dimensions. Let us consider a rectangular element ABCD of an elastic solid with co-ordinates of A as (x, y) and length of sides as Δx and Δy in the natural state.

Let the point A. be defined to a point A′(x +ξ, y +η) then

$$B(x+\Delta x, y) \text{ goes to } B'\left(x+\xi+\Delta x+\frac{\partial \xi}{\partial x}\Delta x,\ y+\eta+\frac{\partial \eta}{\partial x}\Delta x\right)$$

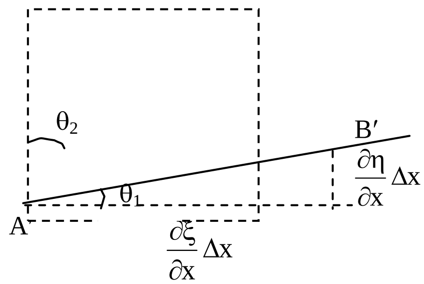


Δy

Δx

A(x,y) B(x+Δx,y)

(Before deformation)



(After deformation)

The point $D(x, y + \Delta y)$ goes to the point

$$D'\left(x + \xi + \frac{\partial \xi}{\partial y}\Delta y,\ \ y + \eta + \Delta y + \frac{\partial \eta}{\partial y}\Delta y\right).$$

Therefore, projected lengths of $A'B'$ along x and y axes are $\Delta x + \frac{\partial \xi}{\partial x}\Delta x$ and $\frac{\partial \eta}{\partial x}\Delta x$

Thus,

$$(A'B')^2 = \left(\Delta x + \frac{\partial \xi}{\partial x}\Delta x\right)^2 + \left(\frac{\partial \eta}{\partial x}\Delta x\right)^2 \qquad (1)$$

The normal strain along x-axis is defined by

$$\in_{xx} = \frac{A'B' - AB}{AB}$$

$$\Rightarrow \qquad A'B' = (1+\in_{xx})\, AB = (1 + \in_{xx})\, \Delta x \qquad |\ AB = \Delta x \qquad (2)$$

From (1) & (2), we have

$$(1+\in_{xx})^2\, (\Delta x)^2 = (\Delta x)^2 \left[\left(1 + \frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial x}\right)^2\right]$$

$$\Rightarrow \qquad (1+\in_{xx})^2 = \left(1 + \frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial x}\right)^2$$

From here, to the first order terms only, we get

$$\in_{xx} = \frac{\partial \xi}{\partial x}.$$

Similarly, the normal strain along the y-axis is

$$\in_{yy} = \frac{\partial \eta}{\partial y}$$

The shearing strain γ_{xy} at the point A is the change in the angle between the sides AB and AD. The right angle $\lfloor\underline{DAB}$ between AB and AD is diminished by $\gamma_{xy} = \theta_1 + \theta_2 = \tan\theta_1 + \tan\theta_2$, θ_1 & θ_2 being small.

i.e. $$\gamma_{xy} = \frac{\frac{\partial \eta}{\partial x}\Delta x}{\left(1+\frac{\partial \xi}{\partial x}\right)\Delta x} + \frac{\frac{\partial \xi}{\partial y}\Delta y}{\left(1+\frac{\partial \eta}{\partial y}\right)\Delta y}$$

$$= \frac{\partial \eta}{\partial x}\left(1+\frac{\partial \xi}{\partial x}\right)^{-1} + \frac{\partial \xi}{\partial y}\left(1+\frac{\partial \eta}{\partial y}\right)^{-1}$$

$$\in_{xy} = \frac{1}{2}(\gamma_{xy}) = \frac{1}{2}\left(\frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y}\right), \qquad \text{upto first order.}$$

We observe that the strains have the nature of change in displacement in a given unit length in a given direction. Hence strain is a tensor of order two.

In the case of fluids, there is no resistance to deformation but only to the time rate of deformation. Hence in fluid dynamics the rate of change of strain with time i.e. rate of strain is to be used in place of strain in elasticity. Thus, for viscous fluids, replacing strains by rates of strain, the corresponding results are obtained to be

$$\in_{xx} = \frac{\partial}{\partial t}\left(\frac{\partial \xi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \xi}{\partial t}\right) = \frac{\partial}{\partial x}(u) = \frac{\partial u}{\partial x}$$

$$\in_{yy} = \frac{\partial v}{\partial y}, v = \frac{\partial \eta}{\partial t}$$

$$\in_{xy} = \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

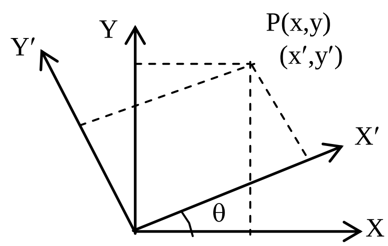
In case of three dimensions, these become

$$\left.\begin{aligned}
\in_{xx} &= \frac{\partial u}{\partial x}, \in_{yy} = \frac{\partial v}{\partial y}, \in_{zz} = \frac{\partial w}{\partial z} \\
\in_{xy} &= \frac{1}{2}(\gamma_{xy}) = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \\
\in_{yz} &= \frac{1}{2}(\gamma_{yz}) = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \\
\in_{zx} &= \frac{1}{2}(\gamma_{zx}) = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)
\end{aligned}\right\} \qquad (A)$$

where u, v, w are the velocity components of the viscous fluid along x, y, z axis respectively.

The six quantities $\in_{xx}$, $\in_{yy}$, $\in_{zz}$, γ_{xy}, γ_{yz}, γ_{zx} in (A) are called components of the rates of strain or **gradients of velocity**

5. Transformation of Rates of Strain. We shall obtain the rates of strain in term of the new co-ordinates x′, y′, changing from x, y to x′, y′. Let us obtain the new axes by rotating the original axes through angle θ and let $l = \cos\theta$, $m = \sin\theta$



Then $\qquad x' = lx + my,\ y' = -mx + ly$

$\Rightarrow \qquad x = lx' - my',\ y = mx' + ly'$

Further, $\qquad \dfrac{\partial}{\partial t}(x') = \dfrac{\partial}{\partial t}(lx + my)$

$\Rightarrow \qquad u' = lu + mv$

and $\qquad v' = -mu + lv$

Also, $\qquad (OP)^2 = x^2 + y^2 = x'^2 + y'^2 \qquad |\because$ they are still perpendicular

Now, $$\in'_{xx} = \frac{\partial u'}{\partial x'} = \left(\frac{\partial u'}{\partial x}\right)\frac{\partial x}{\partial x'} + \left(\frac{\partial u'}{\partial y}\right)\frac{\partial y}{\partial x'}$$

or $$\in'_{xx} = \left(l\frac{\partial u}{\partial x} + m\frac{\partial v}{\partial x}\right)l + \left(l\frac{\partial u}{\partial y} + m\frac{\partial v}{\partial y}\right)m$$

$$= l^2\frac{\partial u}{\partial x} + m^2\frac{\partial v}{\partial y} + lm\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$= l^2 \in_{xx} + m^2 \in_{yy} + lm.\gamma_{xy}$$

Similarly $\quad \in'_{yy} = \dfrac{\partial v'}{\partial y'} = m^2 \in_{xx} + l^2 \in_{yy} - lm\, \gamma_{xy}$

$$\gamma'_{xy} = \frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'} = 2lm\,(\in_{yy} - \in_{xx}) + (l^2 - m^2)\, \gamma_{xy}.$$

which are the rates of strain of the new system in terms of rates of strain in the original system. If we put back $l = \cos\theta$, $m = \sin\theta$, then

$$\left.\begin{aligned} \in'_{xx} &= \frac{\in_{xx} + \in_{yy}}{2} + \frac{\in_{xx} - \in_{yy}}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta \\ \in'_{yy} &= \frac{\in_{xx} + \in_{yy}}{2} - \frac{\in_{xx} - \in_{yy}}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta \\ \in'_{xy} &= \frac{1}{2}(\gamma'_{xy}) = -\frac{\in_{xx} - \in_{yy}}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta \end{aligned}\right\} \qquad (B)$$

These equations give the transformation formulae for the rates of strain.

We observe that the rate of strain is also a tensor of order two, there must exist at least two invariants of the rate of strain to the choice of co-ordinate systems. These can be obtained as follows.

$$\in'_{xx} + \in'_{yy} = (l^2 + m^2)\,(\in_{xx} + \in_{yy})$$

$$= \in_{xx} + \in_{yy} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \operatorname{div} \bar{q}, \qquad \bar{q} = (u, v) \qquad (1)$$

$$\in'_{xx} \in'_{yy} - \frac{(\gamma'_{xy})^2}{4} = (l^2 \in_{xx} + m^2 \in_{yy} + lm\gamma_{xy})\,(m^2 \in_{xx} + l^2 \in_{yy} - lm\,\gamma_{xy})$$

$$- \frac{1}{4}\,[2lm\,(\in_{yy} - \in_{xx}) + (l^2 - m^2)\,\gamma_{xy}]^2$$

$$= (l^4 + 2\,l^2 m^2 + m^4) \in_{xx} \in_{yy} - \frac{\gamma_{xy}^2}{4}\,(l^4 + 2l^2 m^2 + m^4)$$

$$= \in_{xx} \in_{yy} - \frac{\gamma_{xy}^2}{4} \qquad (2)$$

Equation (1) shows that the divergence of the velocity vector at a given point is independent of the orientation of the co-ordinate axes. Equation (2 is related to the dissipation function. i.e. loss of energy due to viscosity.

Let us now consider the general case of the rates of strain in three dimensions. The direction cosines between x, y, z and x′, y′, z′ are related as follows.

	x	y	Z
x′	l_1	m_1	n_1
y′	l_2	m_2	n_2
z′	l_3	m_3	n_3

The relations between co-ordinates in the two systems are

$$x' = l_1x + m_1y \;\; n_1z$$

$$y' = l_2x + m_2y + n_2z$$

$$z' = l_3x + m_3y + n_3z$$

and

$$x = l_1x' + l_2\, y' + l_3\, z'$$

$$y = m_1x' + m_2y' + m_3z'$$

$$z = n_1x' + n_2y' + n_3z'$$

From here, we get

$$u' = l_1u + m_1v + n_1w$$

$$v' = l_2u + m_2v + n_2w$$

$$w' = l_3u + m_3v+ n_3w$$

We shall use these relations to find out the rates of strain w. r. t. the new co-ordinates x′, y′, z′.

Let us work out

$$\in'_{xx} = \frac{\partial u'}{\partial x'} = \frac{\partial u'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial u'}{\partial y}\frac{\partial y}{\partial x'} + \frac{\partial u'}{\partial z}\frac{\partial z}{\partial x'}$$

$$= \left(l_1\frac{\partial u}{\partial x} + m_1\frac{\partial v}{\partial x} + n_1\frac{\partial w}{\partial x}\right)l_1$$

$$+ \left(l_1\frac{\partial u}{\partial y} + m_1\frac{\partial v}{\partial y} + n_1\frac{\partial w}{\partial y}\right)m_1$$

$$+ \left(l_1\frac{\partial u}{\partial z} + m_1\frac{\partial v}{\partial z} + n_1\frac{\partial w}{\partial z}\right)n_1$$

$$= l_1^2 \in_{xx} + m_1^2 \in_{yy} + n_1^2 \in_{zz} + l_1 m_1\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$+ m_1 n_1 \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + n_1 l_1\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

$$= l_1^2 \in_{xx} + m_1^2 \in_{yy} + n_1^2 \in_{zz} + l_1\, m_1\gamma_{xy} + m_1\, n_1\, \gamma_{yz} + n_1 l_1\, \gamma_{zx}$$

Similarly, we have

$$\in'_{yy} = \frac{\partial v'}{\partial y'} = l_2^2 \in_{xx} + m_2^2 \in_{yy} + n_2^2 \in_{zz} + l_2\, m_2\, \gamma_{xy} + m_2 n_2\, \gamma_{yz} + n_2 l_2\, \gamma_{zx}$$

$$\in'_{zz} = \frac{\partial w'}{\partial z'} = l_3^2 \in_{xx} + m_3^2 \in_{yy} + n_3^2 \in_{zz} + l_3 m_3\, \gamma_{xy} + m_3 n_3\, \gamma_{yz} + n_3 l_3\, \gamma_{zx}$$

$$\gamma'_{xy} = \frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'} = \frac{\partial v'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial v'}{\partial y}\frac{\partial y}{\partial x'} + \frac{\partial v'}{\partial z}\frac{\partial z}{\partial x'} + \frac{\partial u'}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial u'}{\partial y}\frac{\partial y}{\partial y'} + \frac{\partial u'}{\partial z}\frac{\partial z}{\partial y'}$$

$$= 2 l_1 l_2 \in_{xx} + 2 m_1\, m_2 \in_{yy} + 2 n_1 n_2 \in_{zz}$$

$$+ (l_1 m_2 + m_1 l_2)\, \gamma_{xy} + (m_1 n_2 + n_1 m_2)\, \gamma_{yz} + (n_1 l_2 + l_1 n_2)\, \gamma_{zx}$$

$$\gamma'_{yz} = \frac{\partial w'}{\partial y'} + \frac{\partial v'}{\partial z'} = 2 l_2\, l_3 \in_{xx} + 2 m_2 m_3 \in_{yy} + 2 n_2 n_3 \in_{zz}$$

$$+ (l_2 m_3 + m_2 l_3)\, \gamma_{xy} + (m_2 n_3 + n_2 m_3)\, \gamma_{yz} + (n_2 l_3 + l_2 n_3)\, \gamma_{zx}$$

$$\gamma'_{zx} = \frac{\partial u'}{\partial z'} + \frac{\partial w'}{\partial x'} = 2l_3 l_1 \in_{xx} + 2m_3\, m_1 \in_{yy} + 2n_3 n_1 \in_{zz}$$

$$+ (l_3 m_1 + m_3 l_1)\, \gamma_{xy} + (m_3 n_1 + n_3 m_1)\, \gamma_{yz} + (n_3 l_1 + l_3 n_1)\, \gamma_{zx}$$

From here, we find

$$\begin{aligned} \in'_{xx} + \in'_{yy} + \in'_{zz} &= (l_1^2 + l_2^2 + l_3^2) \in_{xx} + (m_1^2 + m_2^2 + m_3^2) \in_{yy} \\ &\quad + (n_1^2 + n_2^2 + n_3^2) \in_{zz} + (l_1 m_1 + l_2 m_2 + l_3 m_3)\, \gamma_{xy} \\ &\quad + (m_1 n_1 + m_2 n_2 + m_3 n_3)\, \gamma_{yz} + (n_1 l_1 + n_2 l_2 + n_3 l_3)\, \gamma_{zx} \\ &= \in_{xx} + \in_{yy} + \in_{zz} \end{aligned}$$

where we have used the orthogonality relations

$$l_1^2 + l_2^2 + l_3^2 = 1 \text{ etc}$$

and $\quad l_1 m_1 + l_2 m_2 + l_3 m_3 = 0$ etc.

Thus we conclude that

$$\begin{aligned} \in'_{xx} + \in'_{yy} + \in'_{zz} &= \in_{xx} + \in_{yy} + \in_{zz} \\ &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \text{div } \bar{q} \end{aligned}$$

is invariant.

Similarly,

$$\in'_{xx} \in'_{yy} + \in'_{yy} \in'_{zz} + \in'_{zz} \in'_{xx} - \frac{1}{4} [(\gamma'_{xy})^2 + (\gamma'_{yz})^2 + (\gamma'_{zx})^2]$$

$$= \in_{xx} \in_{yy} + \in_{yy} \in_{zz} + \in_{zz} \in_{xx} - \frac{1}{4} [(\gamma_{xy})^2 + (\gamma_{yz})^2 + (\gamma_{zx})^2]$$

is also invariant.

NOTE. The stress tensor σ_{ij} and the rates of strain $\in_{ij}$ follow the same rules of transformation. Thus, the three equations in (B) can also be written for stress components so that we get the relations between the original and the new stress components as

$$\left.\begin{aligned}
\sigma'_{xx} &= \frac{\sigma_{xx}+\sigma_{yy}}{2}+\frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta+\sigma_{xy}\sin 2\theta \\
\sigma'_{yy} &= \frac{\sigma_{xx}+\sigma_{yy}}{2}-\frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta-\sigma_{xy}\sin 2\theta \\
\sigma'_{xy} &= -\frac{\sigma_{xx}-\sigma_{yy}}{2}\sin 2\theta+\sigma_{xy}\cos 2\theta
\end{aligned}\right\} \qquad (C)$$

6. Relations Between the Stress and Gradients of Velocity

For viscous fluid, the following assumptions are to be made to find the relations between the stress and the rate of strain.

(i) The stress components may be expressed as linear functions of rates of strain components.

(ii) The relations between stress and rates of strain are invariant w.r.t rotation and reflection of co-ordinate axes (symmetry).

(iii) The stress components reduce to the hydrostatic pressure when all the gradients of velocity are zero.

i.e. $\sigma_{xx} = -p = \sigma_{yy} = \sigma_{zz},\ \in_{xx} = \dfrac{\partial u}{\partial x} = 0 = \in_{yy} = \in_{zz}.$

First we consider two dimensional case and then we extend it to three dimensions.

Under the assumption (i), we can write

$$\begin{aligned}
\sigma_{xx} &= A_1 \in_{xx} + B_1 \in_{yy} + C_1\,\gamma_{xy} + D_1 \\
\sigma_{yy} &= A_2 \in_{xx} + B_2 \in_{yy} + C_2\,\gamma_{xy} + D_2 \qquad (1) \\
\sigma_{xy} &= A_3 \in_{xx} + B_3 \in_{yy} + C_3\,\gamma_{xy} + D_3
\end{aligned}$$

where A′s, B′s, C′s and D′s are constants to be determined.

From the assumption (ii), we have

$$\begin{aligned}
\sigma'_{xx} &= A_1 \in'_{xx} + B_1 \in'_{yy} + C_1\,\gamma'_{xy} + D_1 \\
\sigma'_{yy} &= A_2 \in'_{xx} + B_2 \in'_{yy} + C_2\,\gamma'_{xy} + D_2 \qquad (2)
\end{aligned}$$

$$\sigma'_{xy} = A_3 \in'_{xx} + B_3 \in'_{yy} + C_3 \gamma'_{xy} + D_3$$

But the relations between the original and the new stress components are (from equation (C))

$$\left.\begin{aligned}
\sigma'_{xx} &= \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\theta + \sigma_{xy}\sin 2\theta \\
\sigma'_{yy} &= \frac{\sigma_{xx} + \sigma_{yy}}{2} - \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\theta - \sigma_{xy}\sin 2\theta \\
\sigma'_{xy} &= -\frac{\sigma_{xx} - \sigma_{yy}}{2}\sin 2\theta + \sigma_{xy}\cos 2\theta
\end{aligned}\right\} \qquad (3)$$

Using the equation (1) in 1st of (3), we get

$$\begin{aligned}
\sigma'_{xx} &= \frac{1}{2}(A_1 + A_2) \in_{xx} + \frac{1}{2}(B_1+B_2) \in_{yy} + \frac{1}{2}(C_1+C_2)\gamma_{xy} \\
&+ \frac{1}{2}(D_1+D_2) + \frac{1}{2}(A_1-A_2) \in_{xx} \cos 2\theta \\
&+ \frac{1}{2}(B_1-B_2) \in_{yy} \cos 2\theta + \frac{1}{2}(C_1-C_2)\gamma_{xy}\cos 2\theta \\
&+ \frac{1}{2}(D_1-D_2)\cos 2\theta + (A_3 \in_{xx} + B_3 \in_{yy} + C_3\gamma_{xy} + D_3)\sin 2\theta \qquad (4)
\end{aligned}$$

Also, the relations between the original and the new rates of strain are

$$\left.\begin{aligned}
\in'_{xx} &= \frac{\in_{xx} + \in_{yy}}{2} + \frac{\in_{xx} - \in_{yy}}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta \\
\in'_{yy} &= \frac{\in_{xx} + \in_{yy}}{2} - \frac{\in_{xx} - \in_{yy}}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta \\
\gamma'_{xy} &= -\frac{\in_{xx} - \in_{yy}}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta
\end{aligned}\right\} \qquad (5)$$

Using equation (5) in 1st of equations (2), we get

$$\begin{aligned}
\sigma'_{xx} &= \frac{A_1}{2}(\in_{xx} + \in_{yy}) + \frac{A_1}{2}(\in_{xx} - \in_{yy})\cos 2\theta + \frac{A_1}{2}\gamma_{xy}\sin 2\theta \\
&+ \frac{B_1}{2}(\in_{xx} + \in_{yy}) - \frac{B_1}{2}(\in_{xx} - \in_{yy})\cos 2\theta - \frac{B_1}{2}\gamma_{xy}\sin 2\theta
\end{aligned}$$

$$-C_1(\in_{xx} - \in_{yy})\sin 2\theta + C_1\gamma_{xy}\cos 2\theta + D_1 \qquad (6)$$

Comparing co-efficients in (4) & (6), we get

$$\frac{A_1}{2}(1+\cos 2\theta) + \frac{A_2}{2}(1-\cos 2\theta) + A_3\sin 2\theta$$

$$= \frac{A_1}{2}(1+\cos 2\theta) + \frac{B_1}{2}(1-\cos 2\theta) - C_1\sin 2\theta \qquad |\ \in_{xx}$$

$$\frac{B_1}{2}(1+\cos 2\theta) + \frac{B_2}{2}(1-\cos 2\theta) + B_3\sin 2\theta$$

$$= \frac{A_1}{2}(1-\cos 2\theta) + \frac{B_1}{2}(1+\cos 2\theta) + C_1\sin 2\theta \qquad |\ \in_{yy}$$

$$\frac{C_1}{2}(1+\cos 2\theta) + \frac{C_2}{2}(1-\cos 2\theta) + C_3\sin 2\theta$$

$$= \frac{A_1}{2}\sin 2\theta - \frac{B_1}{2}\sin 2\theta + C_1\cos 2\theta \qquad |\ \gamma_{xy}$$

$$\frac{D_1}{2}(1+\cos 2\theta) + \frac{D_2}{2}(1-\cos 2\theta) + D_3\sin 2\theta = D_1$$

From these equations, we get

$$A_2 = B_1 = B(\text{say}),\ B_2 = A_1 = A(\text{say})$$

$$C_2 = A_3 = -C_1 = -B_3 = -C(\text{say})$$

$$C_3 = \frac{A_1 - B_1}{2} = \frac{A-B}{2},\ D_1 = D_2 = D\ (\text{say}),\ D_3 = 0$$

The stress components in terms of the rates of strain are now obtained to be

$$\left.\begin{aligned} \sigma_{xx} &= A\in_{xx} + B\in_{yy} + C\gamma_{xy} + D \\ \sigma_{yy} &= B\in_{xx} + A\in_{yy} - C\gamma_{xy} + D \\ \sigma_{xy} &= -C(\in_{xx} - \in_{yy}) + \frac{A-B}{2}\gamma_{xy} \end{aligned}\right] \qquad (7)$$

To find A, B, C and D, we make use of the assumption that there is symmetry of the fluid about the co-ordinate axes.

Let us take the symmetry w.r.t. the y-axis. If (x_1, y_1) are the new co-ordinates of the point with co-ordinates (x, y), then

$$x_1 = -x,\ y_1 = y$$

i.e. $$u_1 = -u,\ v_1 = v$$

The rates of strain w.r.t. (x_1, y_1) co-ordinates are

$$\in_{x_1x_1} = \frac{\partial u_1}{\partial x_1} = \frac{-\partial u}{\partial x_1} = -\frac{\partial u}{\partial x}\frac{\partial u}{\partial x_1} - \frac{\partial u}{\partial y}\frac{\partial y}{\partial x_1}$$

$$= \frac{\partial u}{\partial x} = \in_{xx} \qquad \Bigg|\ \because \frac{\partial x}{\partial x_1} = -1,\ \frac{\partial y}{\partial x_1} = 0$$

Similarly,

$$\in_{y_1y_1} = \in_{yy},\quad \gamma_{x_1y_1} = -\gamma_{xy}$$

and

$$\sigma_{x_1x_1} = \sigma_{xx},\ \sigma_{y_1y_1} = \sigma_{yy},\ \sigma_{x_1y_1} = -\sigma_{xy}$$

Using these in (7), we get

$$\left.\begin{aligned} \sigma_{x_1x_1} &= A\in_{x_1x_1} + B\in_{y_1y_1} - C\gamma_{x_1y_1} + D \\ \sigma_{y_1y_1} &= B\in_{x_1x_1} + A\in_{y_1y_1} + C\gamma_{x_1y_1} + D \\ \sigma_{x_1y_1} &= C(\in_{x_1x_1} - \in_{y_1y_1}) + \frac{A-B}{2}\gamma_{x_1y_1} \end{aligned}\right] \qquad (8)$$

The relations (7) are invariant where there is a symmetry w.r.t. any co-ordinate transformation and so

$$\left.\begin{aligned} \sigma_{x_1x_1} &= A\in_{x_1x_1} + B\in_{y_1y_1} + C\gamma_{x_1y_1} + D \\ \sigma_{y_1y_1} &= B\in_{x_1x_1} + A\in_{y_1y_1} - C\gamma_{x_1y_1} + D \\ \sigma_{x_1y_1} &= -C(\in_{x_1x_1} - \in_{y_1y_1}) + \frac{A-B}{2}\gamma_{x_1y_1} \end{aligned}\right\} \qquad (9)$$

Comparing (8) & (9), we find C = 0. According to the assumption (iii), we have

$$\sigma_{xx} = \sigma_{yy} = -p,\ \in_{xx} = \in_{yy} = 0$$

Thus from (7), we find $D = -p$, since $C = 0$.

The last equation in (7) becomes

$\sigma_{xy} = \dfrac{A-B}{2}\,\gamma_{xy} = \mu\,\gamma_{xy}$, where $\mu = \dfrac{A-B}{2}$ is called the **co efficient of viscosity.**

The relations in (7) are now,

$$\sigma_{xx} = A\,\epsilon_{xx} + B\,\epsilon_{yy} - p = (A-B)\,\epsilon_{xx} + B\,(\epsilon_{xx} + \epsilon_{yy}) - p$$

$$= 2\mu\,\epsilon_{xx} + B\,\nabla\cdot\bar{q} - p$$

$$\left|\begin{array}{l} \bar{q} = (u, v) \\ \epsilon_{xx} + \epsilon_{yy} = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = \nabla\cdot\bar{q} \end{array}\right.$$

$$\sigma_{yy} = 2\mu\,\epsilon_{yy} + B\nabla\cdot\bar{q} - p.$$

$$\sigma_{xy} = \mu\,\gamma_{xy} = 2\mu\,\epsilon_{xy}$$

These are the required relations between the stress components and the rates of strain in two dimensions.

For three dimensional case, we can write.

$$\left.\begin{aligned} \sigma_{xx} &= 2\mu\epsilon_{xx} + B\nabla\cdot\bar{q} - p = 2\mu\frac{\partial u}{\partial x} + \lambda\nabla\cdot\bar{q} - p \\ \sigma_{yy} &= 2\mu\epsilon_{yy} + B\nabla\cdot\bar{q} - p = 2\mu\frac{\partial v}{\partial y} + \lambda\nabla\cdot\bar{q} - p \\ \sigma_{zz} &= 2\mu\epsilon_{zz} + B\nabla\cdot\bar{q} - p = 2\mu\frac{\partial w}{\partial z} + \lambda\nabla\cdot\bar{q} - p \end{aligned}\right\} \qquad (10)$$

$$\left.\begin{aligned} \sigma_{xy} &= \mu\,\gamma_{xy} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right), \\ \sigma_{yz} &= \mu\,\gamma_{yz} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \\ \sigma_{zx} &= \mu\,\gamma_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \end{aligned}\right\} \qquad (11)$$

where $B \equiv \lambda$.

Also, $\sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 2\mu(\in_{xx} + \in_{yy} + \in_{zz}) + 3\lambda\, \nabla\cdot\bar{q} - 3p$

$$= 2\mu\, \nabla\cdot\bar{q} + 3\lambda\, \nabla\cdot\bar{q} - 3p$$

$$= (2\mu + 3\lambda)\, \nabla\cdot\bar{q} - 3p$$

For incompressible fluid $\nabla\cdot\bar{q} = 0$.

$\Rightarrow$ $\sigma_{xx} + \sigma_{yy} + \sigma_{zz} = -3p$

i.e. $\dfrac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3} = -p$

This shows that the mean normal stress is equal to the hydrostatic pressure (i.e. constant)

NOTE : (i) For compressible fluids, B – = $\lambda \equiv \dfrac{2\mu}{3}$

(ii) Equations (10) and (11) may be combined in tensor form. Thus, if x_i denote the Cartesian co-ordinates, u_i the velocity components (i = 1, 2, 3), then (10) & (11) may be collectively written as

$$\sigma_{ij} = (\lambda\theta - p)\, S_{ij} + \mu(u_{i,j} + u_{j,i}),\ (i, j = 1, 2, 3)$$

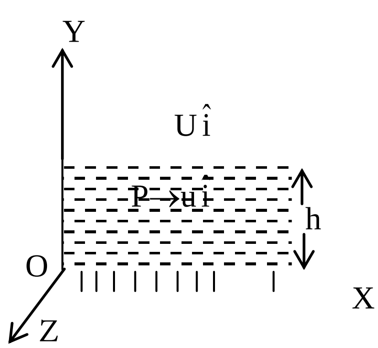
where $\theta = \text{div}\ \bar{q} = u_{j,i},$

$$p = -\frac{1}{3}\sigma_{i,i},\ \theta = 0 \text{ for incompressible flow,}$$

$$\lambda = -\frac{2}{3}\mu \text{ for compressible flow.}$$

(iv) For viscous fluids, stress is linearly proportional to rate of strain. This law is known as Newton's law of viscosity and such fluids are known as **Newtonian fluids**.

7. The Co-efficient of Viscosity and Laminar Flow :



The figure shows two parallel planes y = 0, y = h, a small distance h apart, the space between them being occupied by a thin film of viscous fluid. The plane y = 0 is held fixed and the upper plane is given a constant velocity $U\hat{i}$. If U is not very large, the layers of liquid in contact with y = 0 are at rest and those in contact with y = h are moving with velocity $U\hat{i}$ i.e. there is no slip between fluid and either surface. A velocity gradient is set up in the fluid between the planes. At some point P(x, y, z) in between the planes, the fluid velocity will be $U\hat{i}$, where 0 < u < U and u is independent of x and z. Thus, when y is fixed, u is fixed i.e. fluid moves in layers parallel to two planes. Such flow is termed as **Laminar flow.** Due to viscosity of the fluid there is friction between these layers. Experimental work shows that the shearing stress on the moving plane is proportional to U/h when h is sufficiently small. Thus, we write this stress in the form

$$\sigma_{yx} = \mu \lim_{h\to 0} \frac{U}{h} = \mu \frac{du}{dy}$$

where μ is the co-efficient of viscosity. In aerodynamics, a more important quality is the Kinematic co-efficient of viscosity ν defined by

$$\nu = \mu/\rho.$$

For most fluids μ depends on the pressure and temperature. For gases, according to the Kinetic theory, μ is independent of the pressure but decreases with the temperature.

8. Navier-Stoke's Equations of Motion (Conservation of Linear Momentum)

Let us consider a mass of volume τ enclosed by the surface S in motion at time t. Let dτ be an element of volume, then the mass of this element is ρdτ, ρ being the density of the viscous fluid.

Let the element moves with the velocity $\bar{q}$. The inertial force on the element is

$$\rho d\tau \left(\frac{d\bar{q}}{dt}\right) \qquad \qquad | \bar{F} = m\bar{a}$$

The resultant of inertial forces (or the rate of change of linear momentum) is

$$\overline{F}_I = \iiint \rho \frac{d\overline{q}}{dt} d\tau \qquad (1)$$

Let $\overline{X}$ be the body force per unit mass, then the resultant of body force is

$$\overline{F}_B = \iiint \rho \overline{X} d\tau \qquad (2)$$

The surface force on an element $d\overline{A}$ of the surface is given by the vector

$$\bar{f} = fx\hat{i}_x + fy\hat{i}_y + fz\hat{i}_z$$

$$= (\overline{P}_x.d\overline{A})\hat{i}_x + (\overline{P}_y.d\overline{A})\hat{i}_y + (\overline{P}_z.d\overline{A})\hat{i}_z \qquad (3)$$

where $\hat{i}_x, \hat{i}_y, \hat{i}_z$ are unit vectors, $d\overline{A}$ is the vectorial area of the element and $\overline{P}x, \overline{P}y, \overline{P}z$ are components of stress vector, given by

$$\left.\begin{aligned} \overline{P}_x &= \sigma_{xx}\,\hat{i}_x + \sigma_{xy}\,\hat{i}_y + \sigma_{xz}\,\hat{i}_z \\ \overline{P}_y &= \sigma_{yx}\,\hat{i}_x + \sigma_{yy}\,\hat{i}_y + \sigma_{yz}\,\hat{i}_z \\ \overline{P}_z &= \sigma_{zx}\,\hat{i}_x + \sigma_{zy}\,\hat{i}_y + \sigma_{zz}\,\hat{i}_z \end{aligned}\right] \qquad | \; T_i^x = \tau_{ij}\, x_j \qquad (4)$$

The resultant of the surface forces is given by

$$\overline{F}_S = \hat{i}_x \iint \overline{P}_x.d\overline{A} + \hat{i}_y \iint \overline{P}_y.d\overline{A} + \hat{i}_z \iint \overline{P}_z.d\overline{A} \qquad (5)$$

Using Gauss divergence theorem this can be written as

$$\overline{F}_S = \hat{i}_x \iiint \nabla\cdot\overline{P}_x d\tau + \hat{i}_y \iiint \nabla\cdot\overline{P}_y d\tau + \hat{i}_z \iiint \nabla\cdot\overline{P}_z d\tau \quad (6) \quad \Big| \because d\overline{A} = \hat{n}\, dS$$

Let us use the law of conservation of momentum. By this law, the time rate of change of linear momentum is equal to the total force on the fluid mass. Equating the resultant of body and surface forces with that of inertial forces, we obtain.

$$\iiint \rho \frac{d\overline{q}}{dt} d\tau = \iiint \rho \overline{X} d\tau + \hat{i}_x \iiint \nabla\cdot\overline{P}_x d\tau + \hat{i}_y \iiint \nabla\cdot\overline{P}_y d\tau + \hat{i}_z \iiint \nabla\cdot\overline{P}_z d\tau \qquad (7)$$

Since $d\tau$ is an arbitrary volume element, so we have

$$\rho \frac{d\overline{q}}{dt} = \rho\overline{X} + \nabla\cdot\overline{P}_x\,\hat{i}_x + \nabla\cdot\overline{P}_y\,\hat{i}_y + \nabla\cdot\overline{P}_z\,\hat{i}_z \qquad (8)$$

This is the required equation of motion in vector form using the values of $\overline{P}_x, \overline{P}_y, \overline{P}_z,$ we get

$$\nabla \cdot \overline{P}_x = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z}$$

$$\nabla \cdot \overline{P}_y = \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z}$$

$$\nabla \cdot \overline{P}_x = \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}$$

and let $\overline{q} = (u, v, w)$, $\overline{X} = (X_x, X_y, X_z)$ then the equations of motion can be put as

$$\left.\begin{aligned}
\rho \frac{du}{dt} &= \rho X_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} \\
\rho \frac{dv}{dt} &= \rho X_y + \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} \\
\rho \frac{dw}{dt} &= \rho X_z + \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}
\end{aligned}\right\} \qquad (9)$$

These are the equations of motion in terms of the stress components. (We have also drawn these equations previously)

Also, we know that

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \overline{q} \cdot \nabla \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

and the relations between stress and rates of strain are

$$\sigma_{xx} = 2\mu \frac{\partial u}{\partial x} + \lambda \nabla \cdot \overline{q} - p$$

$$\sigma_{yy} = 2\mu \frac{\partial v}{\partial y} + \lambda \nabla \cdot \overline{q} - p$$

$$\sigma_{zz} = 2\mu \frac{\partial w}{\partial z} + \lambda \nabla \cdot \overline{q} - p$$

$$\sigma_{xy} = \mu \gamma_{xy} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$$

$$\sigma_{yz} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right),$$

$$\sigma_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

Using these in (9), we get

$$\left.\begin{aligned}
\rho\frac{du}{dt} &= \rho X_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial u}{\partial x} - \frac{2}{3}\nabla\cdot\overline{q}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right]\\
\rho\frac{dv}{dt} &= \rho X_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial y}\left[\mu\left(2\frac{\partial v}{\partial y} - \frac{2}{3}\nabla\cdot\overline{q}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right]\\
\rho\frac{dw}{dt} &= \rho X_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial z}\left[\mu\left(2\frac{\partial w}{\partial z} - \frac{2}{3}\nabla\cdot\overline{q}\right)\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\right]
\end{aligned}\right\}$$

(10)

where $\lambda = -\frac{2\mu}{3}$ compressible fluids.

The equation in (10) are called Navier-Stoke's equations for a viscous **compressible** fluid.

NOTE (i) If μ = co-efficient of viscosity = constant, then Navier-Stoke's equations (10) become

$$\rho\frac{du}{dt} = \rho X_x - \frac{\partial p}{\partial x} + \frac{1}{3}\mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \mu\nabla^2 u$$

$$\rho\frac{du}{dt} = \rho X_x - \frac{\partial p}{\partial x} + \frac{1}{3}\mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \mu\nabla^2 u$$

$$\rho\frac{dw}{dt} = \rho X_z - \frac{\partial p}{\partial z} + \frac{1}{3}\mu\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \mu\nabla^2 w$$

which can be expression in vector form as

$$\rho\frac{d\overline{q}}{dt} = \rho\left[\frac{\partial\overline{q}}{\partial t} + (\overline{q}\cdot\nabla)\overline{q}\right] = \rho\overline{X} - \nabla p + \mu\nabla^2\overline{q} + \frac{\mu}{3}\nabla(\nabla\cdot\overline{q})$$

(ii) For incompressible fluid, ρ = constant,

$$\mu = \text{constant}, \qquad \nabla \cdot \bar{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

Thus the equations become

$$\frac{d\bar{q}}{dt} = \frac{\partial \bar{q}}{\partial t} + (\bar{q} \cdot \nabla)\bar{q} = \bar{X} - \frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2 \bar{q}$$

i.e.
$$\frac{d\bar{q}}{dt} = \bar{X} - \frac{\nabla p}{\rho} + \nu \nabla^2 \bar{q}$$

where $\nu = \mu/\rho$ is called the Kinematic co-efficient of viscosity.

For steady motion with no body forces, we have

$$(\bar{q} \cdot \nabla)\bar{q} = \frac{-\nabla p}{\rho} + \frac{\mu}{\rho}\nabla^2 \bar{q} \qquad \left| \; \frac{\partial \bar{q}}{\partial t} = 0, \; \bar{X} = 0 \right.$$

(iii) If there is no shear at all i.e $\mu = 0$, then

$$\frac{d\bar{q}}{dt} = \frac{\partial \bar{q}}{\partial t} + (\bar{q} \cdot \nabla)\bar{q} = \bar{X} - \frac{\nabla p}{\rho}$$

These are Euler's equations for an incompressible non-viscous fluid.